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Classical Skew Orthogonal Polynomials and Random Matrices

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Skew orthogonal polynomials arise in the calculation of the n-point distribution function for the eigenvalues of ensembles of random matrices with orthogonal or symplectic symmetry. In particular, the distribution functions are completely determined by a certain sum involving the skew orthogonal polynomials. In the case that the eigenvalue probability density function involves a classical weight function, explicit formulas for the skew orthogonal polynomials are given in terms of related orthogonal polynomials, and the structure is used to give a closed-form expression for the sum. This theory treates all classical cases on an equal footing, giving formulas applicable at once to the Hermite, Laguerre, and Jacobi cases.

KEY WORDS: Random matrices; correlation functions; orthogonal polynomials.

1. INTRODUCTION

The classical polynomials play an essential role in calculating statistical properties of the eigenvalues of certain classes of random matrices with normally distributed elements. As a concrete example, consider a random $N \times N$ Hermitian matrix X in which the diagonal elements (which must be real) and the upper triangular elements are independently chosen with normal distributions $\mathrm{N}[0, 1/\sqrt{2}]$ and $\mathrm{N}[0, 1/2] + i\mathrm{N}[0, 1/2]$ respectively.

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Then the corresponding eigenvalue probability density function (p.d.f.) is proportional to

$$\prod_{j=1}^{N} w_2(x_j) \prod_{1 \leqslant j<k \leqslant N} (x_k - x_j)^2, \qquad w_2(x) = e^{-x^2} \tag{1.1}$$

Note that this involves the classical weight function e^{-x^2}.

The monic orthogonal polynomials corresponding to the weight function $w_2(x)$, $\{p_j(x)\}$ say, are introduced by writing the product of differences in (1.1) as the square of the Vandermonde determinant, and adding together appropriate multiples of the columns. This shows that (1.1) is equal to

$$\prod_{j=1}^{N} w_2(x_j)(\det[p_{j-1}(x_k)]_{j,\,k=1,\dots,\,N})^2 \tag{1.2}$$

This form of the p.d.f. provides the starting point for the calculation of the n-point distribution functions (see Section 3), which in turn are used in calculating spacing distributions (see e.g., ref. 9).

The p.d.f. (1.1) with the classical weight function $w_2(x) = x^a e^{-x}$, $(x > 0)$, and $w_2(x) = x^a(1-x)^b$, $(0 < x < 1)$, also occurs in random matrix problems (see e.g., ref. 10). Thus if X is a rectangular $n \times m$, $(n \geqslant m)$, matrix with complex elements independently distributed according to the normal distribution $\mathrm{N}[0, 1/\sqrt{2}] + i\mathrm{N}[0, 1/\sqrt{2}]$, then the eigenvalues of $X^\dagger X$ have the distribution (1.1) with $w_2(x) = x^{n-m}e^{-x}$ and $N = m$. Similarly, if X_1 and X_2 are rectangular matrices of dimensions $n_1 \times m$ and $n_2 \times m$, then the eigenvalues of $X_1^\dagger X_1(1 + X_2^\dagger X_2)^{-1}$ have the distribution (1.1) with $w_2(x) = x^{n_1-m}(1-x)^{n_2-m}$ and $N = m$.

If the random matrices X which lead to the eigenvalue p.d.f (1.1) with the classical weight functions noted above are specified to have real elements, or real quaternion elements represented by 2×2 blocks of the form

$$\begin{bmatrix} z_{jk} & w_{jk} \\ -\bar{w}_{jk} & \bar{z}_{jk} \end{bmatrix}$$

(in which case the eigenvalues are doubly degenerate), the corresponding eigenvalue p.d.f. is then proportional to

$$\prod_{j=1}^{N} w_\beta(x_j) \prod_{1 \leqslant j<k \leqslant N} |x_k - x_j|^\beta \tag{1.3}$$

Here $\beta = 1$ when the elements are real, while $\beta = 4$ when the elements are real quaternion, and the weight function is again of the same classical form as in the complex case (1.1) (up to scaling of the eigenvalues x_j). We remark that ensembles of such matrices have orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) symmetry. Instead of writing the product of differences in (1.3) in terms of an ordinary determinant involving skew orthogonal polynomials, it has been shown by Mahoux and Mehta[8] (see also refs. 13, 7, and 14) that for the purposes of calculating the n-point distribution one should introduce a quaternion determinant involving skew orthogonal polynomials. The definition of a quaternion determinant can be found for example in ref. 5; we remark here that for the $2N \times 2N$ matrices Q occuring in such a calculation (which technically are self dual) the quaternion determinant, to be denoted Tdet, is related to the usual Pfaffian via the formula

$$\text{Tdet}\, Q = \text{Pf}(ZQ), \qquad Z := 1_N \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.4}$$

Our interest is in the calculation of the classical skew orthogonal polynomials in terms of classical orthogonal polynomials, and the application of these formulas in the evaluation of the matrix elements in the Tdet expression for the n-point distribution. We begin in Section 2 by revising the relationship between the p.d.f. (1.3) for $\beta = 1$ and 4 and skew orthogonal polynomials. Also revised is the corresponding expression for the n-point distribution functions in terms of the skew orthogonal polynomials. Next we present the recent theory of Adler and van Moerbeke[2] which identifies the family of orthogonal polynomials naturally related to a particular family of skew orthogonals, and furthermore relates the skew orthogonal polynomials at $\beta = 1$ to those at $\beta = 4$.

In Section 3 this theory is used to calculate the classical skew orthogonal polynomials in terms of their natural basis of orthogonal polynomials. The results are shown to be consistent with expressions obtained in the earlier studies of Nagao and Wadati.[13] Application of these expansion formulas is made in Section 4, in which the fundamental series involving the skew orthogonal polynomials occuring in the expression of the n-point distribution is summed for all the classical ensembles. In the Hermite case this reclaims known results. For the Laguerre ensemble our summation formula differs from that obtained in recent studies of Widom[15] and Forrester *et al.*[16]—the reconciliation of the two seemingly different forms is made in the Appendix. The remaining classical case is the Jacobi ensemble. Here our summation formulas constitute new results.

2. REVISION

2.1. Skew Orthogonal Polynomials and the *n*-Point Distribution

The *n*-point distribution $\rho(n)$ with respect to the p.d.f. (1.3) is defined as a ratio of multiple integrals according to

$$\rho_{(n)}(x_1,..., x_n) = \prod_{j=1}^{n} w_\beta(x_j) \frac{\prod_{l=n+1}^{N} \int_{-\infty}^{\infty} dx_l \, w_\beta(x_l) \prod_{1 \leqslant j<k \leqslant N} |x_k - x_j|^\beta}{\prod_{l=1}^{N} \int_{-\infty}^{\infty} dx_l \, w_\beta(x_l) \prod_{1 \leqslant j<k \leqslant N} |x_k - x_j|^\beta} \tag{2.1}$$

Generalizing the pioneering work of Dyson,[5] it has been shown by Mahoux and Mehta[8] that for $\beta = 1$ and $\beta = 4$ the distribution $\rho_{(n)}$ can be evaluated as an $n \times n$ quaternion determinant involving certain skew orthogonal polynomials.

In preparation for presenting these results, we recall that an inner product $\langle f, g \rangle$ is referred to as skew if

$$\langle f, g \rangle = -\langle g, f \rangle$$

while a family of (monic) polynomials $\{q_n(x)\}_{n=0, 1, 2...}$ are said to be skew orthogonal if

$$\begin{aligned} \langle q_{2m}, q_{2n+1} \rangle &= -\langle q_{2n+1}, q_{2m} \rangle = r_m \, \delta_{m, n} \\ \langle q_{2m}, q_{2n} \rangle &= \langle q_{2m+1}, q_{2n+1} \rangle = 0 \end{aligned} \tag{2.2}$$

Equivalently, for any given N we require

$$[\langle q_j, q_k \rangle]_{j, k=0,..., 2N-1} = R$$

$$R := \begin{bmatrix} 0 & r_1 & & & & & \\ -r_0 & 0 & & & & & \\ & & 0 & r_1 & & & \\ & & -r_1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & r_{N-1} \\ & & & & & -r_{N-1} & 0 \end{bmatrix} \tag{2.3}$$

We remark that the relations (2.2) are unchanged under the replacement

$$q_{2m+1}(x) \mapsto q_{2m+1}(x) + \alpha_{2m} q_{2m}(x) \tag{2.4}$$

for arbitrary α_{2m}, so the skew orthogonal transformations are non-unique up to this mapping.

The specific skew inner products of relevance to the calculation of (2.3) are

$$\langle f, g\rangle_1 := \tfrac{1}{2} \int_{-\infty}^{\infty} dx\, e^{-V(x)} \int_{-\infty}^{\infty} dy\, e^{-V(y)} \operatorname{sgn}(y-x)\, f(x)\, g(y) \tag{2.5}$$

and

$$\langle f, g\rangle_4 := \tfrac{1}{2} \int_{-\infty}^{\infty} dx\, e^{-2V(x)}(f(x)\, g'(x) - f'(x)\, g(x)) \tag{2.6}$$

for $\beta = 1$ and $\beta = 4$ respectively, where in (2.3) we have chosen

$$w_1(x) = e^{-V(x)} \qquad w_4(x) = e^{-2V(x)} \tag{2.7}$$

We denote the corresponding monic skew orthogonal polynomials by $\{q_j^{(1)}\}_{j=0,1,\dots}$ and $\{q_j^{(4)}\}_{j=0,1,\dots}$ and apply similar superscripts to the matrix R and its elements.

With these preliminaries, the quaternion determinant formulas for (2.1) can now be presented.

$\beta = 1$, N **even**

For $\beta = 1$ the result depends on the parity of N. Let N be even and set

$$\begin{aligned} \Phi_k(x) &:= \tfrac{1}{2} \int_{-\infty}^{\infty} \operatorname{sgn}(x-y)\, q_k^{(1)}(y)\, e^{-V(y)}\, dy \\ f_1(x, y) &= \begin{bmatrix} S_1(x, y) & I_1(x, y) \\ D_1(x, y) & S_1(y, x) \end{bmatrix} \end{aligned} \tag{2.8}$$

with

$$\begin{aligned} S_1(x, y) &= \sum_{k=0}^{N/2-1} \frac{e^{-V(y)}}{r_k^{(1)}} (\Phi_{2k}(x)\, q_{2k+1}^{(1)}(y) - \Phi_{2k+1}(x)\, q_{2k}^{(1)}(y)) \\ D_1(x, y) &= \frac{\partial}{\partial x} S_1(x, y) \\ I_1(x, y) &= \frac{1}{2} \int_{-\infty}^{\infty} S_1(x, z) \operatorname{sgn}(z-y)\, dz - \frac{1}{2} \operatorname{sgn}(x-y) \end{aligned} \tag{2.9}$$

Then with $w_1(x)=e^{-V(x)}$ in (2.1) we have[8]

$$\rho_{(n)}(x_1,..., x_n)=\mathrm{qdet}[f_1(x_j, x_k)]_{j,k=1,...,n} \tag{2.10}$$

$\beta=1$, N **odd**

Let $\{q_n^{(1)}\}_{n=0,1,...}$ and $\{r_n^{(1)}\}_{n=0,...,(N-1)/2-1}$ be as in the N even case, and define

$$r^{(1)}_{(N-1)/2} := \frac{1}{2}\int_{-\infty}^{\infty} dx\, e^{-V(x)} q^{(1)}_{N-1}(x)$$

$$\hat{q}^{(1)}_n(x) := q^{(1)}_n(x) - \frac{1}{2r^{(1)}_{(N-1)/2}}\left(\int_{-\infty}^{\infty} dx'\, e^{-V(x')} q^{(1)}_n(x')\right) q^{(1)}_{N-1}(x)$$

$$(n=0,..., N-2)$$

$$\hat{q}^{(1)}_{N-1}(x) := q^{(1)}_{N-1}(x)$$

$$\hat{\Phi}_n(x) := \frac{1}{2}\int_{-\infty}^{\infty} dy\, e^{-V(y)}\, \mathrm{sgn}(x-y)\, \hat{q}^{(1)}_n(x) \qquad (n=0,..., N-1)$$

$$f_1^{\mathrm{odd}} = \begin{bmatrix} S_1^{\mathrm{odd}}(x, y) & I_1^{\mathrm{odd}}(x, y) \\ D_1^{\mathrm{odd}}(x, y) & S_1^{\mathrm{odd}}(y, x) \end{bmatrix}$$

$$S_1^{\mathrm{odd}}(x, y) = \sum_{k=0}^{(N-1)/2-1} \frac{e^{-V(y)}}{r_k^{(1)}} \left(\hat{\Phi}_{2k}(x)\, \hat{q}^{(1)}_{2k+1}(y) - \hat{\Phi}_{2k+1}(x)\, \hat{q}^{(1)}_{2k}(y)\right) + \frac{\hat{q}^{(1)}_{N-1}(y)}{2r^{(1)}_{(N-1)/2}}$$

$$D_1^{\mathrm{odd}}(x, y) = \frac{\partial}{\partial x} S_1^{\mathrm{odd}}(x, y)$$

$$I_1^{\mathrm{odd}}(x, y) = \frac{1}{2}\int_{-\infty}^{\infty} S_1^{\mathrm{odd}}(x, z)\, \mathrm{sgn}(z-y)\, dz - \frac{1}{2}\mathrm{sgn}(x-y) + \frac{\hat{\Phi}_{N-1}(x)}{2r^{(1)}_{(N-1)/2}} \tag{2.11}$$

With this notation, (2.10) applies with f_1 replaced by f_1^{odd}.[7]

$\beta = 4$

For $\beta = 4$ the results are independent of the parity of N. Let

$$S_4(x, y) = \sum_{m=0}^{N-1} \frac{e^{-V(x)}}{2r_m^{(4)}} \left(q_{2m}^{(4)}(x) \frac{d}{dy} (e^{-V(y)} q_{2m+1}^{(4)}(y)) - q_{2m+1}^{(4)}(x) \frac{d}{dy} (e^{-V(y)} q_{2m}^{(4)}(y)) \right)$$

$$D_4(x, y) = \frac{\partial}{\partial x} S_4(x, y) \tag{2.12}$$

$$I_4(x, y) := -\int_x^y S_4(x, y')\, dy'$$

and set

$$f_4(x, y) := \begin{bmatrix} S_4(x, y) & I_4(x, y) \\ D_4(x, y) & S_4(y, x) \end{bmatrix}$$

Then with $w_4(x) = e^{-2V(x)}$ in (2.1) we have,[13]

$$\rho_{(n)}(x_1, ..., x_n) = \mathrm{Tdet}[f_4(x_j, x_k)]_{j, k = 1, ..., n} \tag{2.13}$$

2.2. Construction of the Skew Orthogonal Polynomials

A number of works have addressed the construction of the skew orthogonal polynomials.[8, 13, 4, 2] In particular Nagao and Wadati[13] have succeeded in giving explicit formulas for the classical skew orthogonal polynomials in terms of related orthogonal polynomials. More recently Adler and van Moerbeke,[2] generalizing some earlier work of Brézin and Neuberger,[4] have identified a natural orthogonal polynomial basis for the skew orthogonal polynomials $\{q_n(x)\}$ corresponding to the skew symmetric inner products (2.5) and (2.6) and shown how to use this basis to specify the $q_n(x)$. Here we will revise this formalism, which in the next section will be implemented in the classical cases and the results compared with those known from the work of Nagao and Wadati.[13]

Following ref. 2, with $V(x)$ as in (2.7), let

$$2V'(x) = \frac{g(x)}{f(x)} \tag{2.14}$$

be a rational function, and introduce the operator

$$\mathbf{n} := f\,\frac{d}{dx} + \left(\frac{f' - g}{2}\right) = \sqrt{e^{2V} f}\,\frac{d}{dx}\,\sqrt{f e^{-2V}} \tag{2.15}$$

Furthermore, introduce the symmetric inner product

$$(\phi, \psi)_2 := \int_{-\infty}^{\infty} e^{-2V(x)}\,\phi(x)\,\psi(x)\,dx \tag{2.16}$$

As was noted in ref. 1, the operator $\mathbf{n}$ is skew symmetric with respect to this inner product:

$$(\phi, \mathbf{n}\psi)_2 = -(\mathbf{n}\phi, \psi)_2$$

Then it is shown in ref. 2 that

$$\begin{aligned} (\phi, \mathbf{n}^{-1}\psi)_2 &= -\langle \phi, \psi \rangle_1 \big|_{2V(x) \mapsto 2V(x) + \log f(x)} \\ (\phi, \mathbf{n}\psi)_2 &= \langle \phi, \psi \rangle_4 \big|_{2V(x) \mapsto 2V(x) - \log f(x)} \end{aligned} \tag{2.17}$$

where $\mathbf{n}^{-1}$ can be represented in terms of an integral operator by its action

$$\mathbf{n}^{-1}\psi[x] = (f(x)\,e^{-2V(x)})^{-1/2}\,\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-V(y)}\,\mathrm{sgn}(x-y)(f(y))^{-1/2}\,\psi(y)\,dy$$

Let $\{p_j(x)\}_{j=0,1,\ldots}$ denote the monic orthogonal polynomials corresponding to the symmetric inner product (2.16). Searching for orthogonal polynomials is tantamount to factorizing the (symmetric) moment matrix (associated with the inner product (2.16)) into the product of a lower times an upper triangular matrix. This is possible, because all $n \times n$ upper left hand corners of the moment matrix are non singular. The entries of the lower triangular matrix then provides the coefficients of the orthogonal polynomials.

However, for skew symmetric inner products, the semi infinite moment matrix is skew symmetric and thus all the odd upper left hand corners are singular. But still, such a matrix allows a factorization as before, only upon inserting a matrix consisting of 2×2 skew symmetric blocks along the diagonal and otherwise zero. Here again, the entries of the lower triangular matrix are the coefficients of the skew orthogonal polynomials. For more details, see ref. 3.

The practicality of the formulas (2.17) lies in the fact that with the monic orthogonal polynomials $\{p_j(x)\}_{j=0,1,\ldots}$, introduced above, the matrix $[(p_j, \mathbf{n}p_k)_2]_{j,k=0,\ldots,N-1}$ (which by the formula preceding (2.17) is skew symmetric) has non-zero terms only in a finite number of diagonals about the main diagonal for f and g polynomials in (2.14). In particular, for the classical polynomials this number is precisely two, giving

$$\mathcal{N} := [(p_j, \mathbf{n}p_k)_2]_{j,k=0,\ldots,N-1} = \begin{bmatrix} 0 & c_0 & 0 & 0 & \cdots & & \\ -c_0 & 0 & c_1 & 0 & \cdots & & \\ 0 & -c_1 & 0 & c_2 & \cdots & & \\ 0 & 0 & -c_2 & 0 & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ & & & & & 0 & c_{N-1} \\ & & & & & -c_{N-1} & 0 \end{bmatrix} \tag{2.18}$$

Equivalently, for $\{p_j(x)\}_{j=0,1,\ldots}$ a set of classical orthogonal polynomials,

$$\mathbf{n}p_k(x) = -\frac{c_k}{(p_{k+1}, p_{k+1})_2}\, p_{k+1}(x) + \frac{c_{k-1}}{(p_{k-1}, p_{k-1})_2}\, p_{k-1}(x) \tag{2.19}$$

The simple structure of (2.18) allows the classical skew orthogonal polynomials to be expressed in terms of their orthogonal polynomial counterparts. Let $\{\tilde{q}_j^{(1)}(x)\}$ and $\{\tilde{q}_j^{(4)}(x)\}$ denote the skew orthogonal polynomials corresponding to the inner products (2.5) and (2.6) with $V(x) \mapsto V(x) + \log f(x)$ and $V(x) \mapsto V(x) - \log f(x)$ respectively. Similarly let $\{\tilde{r}_j^{(1)}\}$ and $\{\tilde{r}_j^{(4)}\}$ denote the corresponding normalizations as in (2.3). We can write the matrix $[(p_j, \mathbf{n}^{-1}p_k)_2]_{j,k=0,\ldots,N-1}$ in terms of $\mathcal{N}$. Thus with

$$D := \mathrm{diag}((p_j, p_j)_2)_{j=0,\ldots,N-1} \tag{2.20}$$

we have

$$([(p_j, \mathbf{n}^{-1}p_k)_2]_{j,k=0,\ldots,N-1}\, D^{-1}\mathcal{N})_{jk} = -\sum_{l=0}^{N-1} \frac{(\mathbf{n}^{-1}p_j, p_l)_2}{(p_l, p_l)_2}\, (p_l, \mathbf{n}p_k)_2 \tag{2.21}$$

Now according to (2.19)

$$\sum_{l=0}^{N-1} p_l(x)\, \frac{(p_l, \mathbf{n}p_k)_2}{(p_l, p_l)_2} = \mathbf{n}p_k(x) \qquad \text{for} \quad k \neq N-1$$

Substituting this back in (2.21) and using the fact that $\mathbf{n}^{-1}$ is the inverse of $\mathbf{n}$ shows

$$[(p_j, \mathbf{n}^{-1} p_k)_2]_{j,\,k=0,\dots,\,N-1}\, D^{-1}\mathcal{N} = D' \tag{2.22}$$

where $D' = D + C_k$ with C_k having all entries zero except for the final column (the explicit value of these entries will not be required).

Next introduce transition matrices $Q^{(1)}$ and $Q^{(4)}$ such that

$$[\tilde{q}_j^{(s)}(x)]_{j=0,\dots,\,N-1} = Q^{(s)}[p_j(x)]_{j=0,\dots,\,N-1}, \qquad s = 1 \text{ or } 4 \tag{2.23}$$

Both these transition matrices must be lower triangular matrices with 1's along the diagonal. Then we have

$$\begin{aligned}\tilde{R}^{(1)} &:= [\langle \tilde{q}_j^{(1)}, \tilde{q}^{(1)}{}_k\rangle_1 |_{V(x)\mapsto V(x)+(1/2)\log f(x)}]_{j,\,k=0,\dots,\,N-1}\\ &= Q^{(1)}[\langle p_j, p_k\rangle_1 |_{V(x)\mapsto V(x)+(1/2)\log f(x)}]_{j,\,k=0,\dots,\,N-1}\, Q^{(1)\,T}\\ &= -Q^{(1)}[(p_j, \mathbf{n}^{-1}p_k)_2]_{j,\,k=0,\dots,\,N-1}\, Q^{(1)\,T} = -Q^{(1)}D'\mathcal{N}^{-1}DQ^{(1)\,T}\end{aligned} \tag{2.24}$$

where in obtaining the first equality of the final line the second formula of (2.17) has been used, while in the final equality (2.22) has been used. Taking inverses of both sides of (2.24) shows that it is equivalent to the statement that

$$Q^{(1)\,T}\tilde{R}^{(1)-1}Q^{(1)} = -D^{-1}\mathcal{N}D'^{-1} \tag{2.25}$$

A similar calculation shows

$$\tilde{R}^{(4)} := [\langle \tilde{q}_j^{(4)}, \tilde{q}_k^{(4)}\rangle_4 |_{V(x)\mapsto V(x)-(1/2)\log f(x)}]_{j,\,k=0,\dots,\,N-1} = Q^{(4)}\mathcal{N}Q^{(4)\,T} \tag{2.26}$$

or equivalently

$$Q^{(4)-1}\tilde{R}^{(4)}(Q^{(4)\,T})^{-1} = \mathcal{N} \tag{2.27}$$

To conclude, $\mathcal{N}^{-1}$ and $\mathcal{N}$ are skew-symmetric matrices and therefore admit, as was noted earlier, a factorization into the product of lower and upper triangular matrices, only upon inserting a matrix of type R. The decompositions (2.25) and (2.27) are precisely of that nature, after insertion of the matrices $\tilde{R}^{(1)}$ and $\tilde{R}^{(4)}$ respectively:

$$\begin{aligned}\mathcal{N}^{-1} &= -D'^{-1}Q^{(1)-1}\tilde{R}^{(1)}(q^{(1)\,T})^{-1}\, D^{-1} &\qquad (\beta = 1)\\ \mathcal{N} &= Q^{(4)-1}\tilde{R}^{(4)}(Q^{(4)\,T})^{-1} &\qquad (\beta = 4)\end{aligned}$$

3. THE CLASSICAL SKEW ORTHOGONAL POLYNOMIALS

3.1. Summary of Known Results

Hermite Case. Here $e^{-V(x)} = e^{-x^2/2}$. The monic orthogonal polynomials with respect to the inner product (2.16), and corresponding normalization are then

$$\begin{aligned} p_k(x) &= 2^{-k} H_k(x) \\ (p_k, p_k)_2 &= \pi^{1/2} 2^{-k} k! \end{aligned} \tag{3.1}$$

The skew orthogonal polynomials and corresponding normalizations can be expressed in terms of these orthogonal polynomials according to ref. 8

$$\begin{aligned} q_{2m}^{(1)}(x) &= p_{2m}(x) \\ q_{2m+1}^{(1)}(x) &= p_{2m+1}(x) - m p_{2m-1}(x) \\ r_n^{(1)} &= 2^{-2n} \sqrt{\pi}\, \Gamma(2n+1) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} q_{2m+1}^{(4)}(x) &= p_{2m+1}(x) \\ q_{2m}^{(4)}(x) &= m! \sum_{n=0}^{m} \frac{1}{n!} p_{2n}(x) \\ r_n^{(4)} &= 2^{-2n-1} \sqrt{\pi}\, \Gamma(2n+2) \end{aligned} \tag{3.3}$$

Laguerre Case. Here we have

$$\begin{aligned} e^{-V(x)} &= x^{a/2} e^{-x/2} \\ p_k(x) &= (-1)^k\, k!\, L_k^a(x) \\ (p_k, p_k)_2 &= \Gamma(k+1)\, \Gamma(a+k+1) \end{aligned} \tag{3.4}$$

where L_k^a denotes the Laguerre polynomial. With

$$e^{-V(x)} \mapsto e^{-V(x) - (1/2)\log f(x)} = x^{(a-1)/2} e^{-x/2} \tag{3.5}$$

in (2.5) the corresponding monic skew orthogonal polynomials and normalization are[13]

$$\begin{aligned}\tilde{q}^{(1)}_{2m}(x) &= p_{2m}(x)\\ \tilde{q}^{(1)}_{2m+1}(x) &= -(2m+1)!\, L^{a-1}_{2m+1}(x) - (2m)!\,(a+2m)\frac{d}{dx} L^{a}_{2m}(x)\\ &= p_{2m+1}(x) + (2m+1)\, p_{2m}(x) - 2m(a+2m)\, p_{m-1}(x)\\ \tilde{r}^{(1)}_{n} &= 2\Gamma(n+1)\,\Gamma(a+2n+2)\end{aligned} \tag{3.6}$$

where in obtaining the second equality in (3.6) use has been made of the formulas

$$L^{a-1}_{n}(x) = L^{a}_{n}(x) - L^{a}_{n-1}(x) \qquad \frac{d}{dx} L^{a}_{n}(x) = -L^{a+1}_{n-1}(x) \tag{3.7}$$

Recalling the fact that the skew orthogonal polynomials are non-unique up to the transformation (2.4), we see that we can equally as well write in place of the second formula in (3.6)

$$\tilde{q}^{(1)}_{2m+1}(x) = p_{2m+1}(x) - 2m(a+2m)\, p_{2m-1}(x) \tag{3.8}$$

At $\beta = 4$ put

$$e^{-2V(x)} \mapsto e^{-2V(x) + \log f(x)} = x^{a+1} e^{-x} \tag{3.9}$$

in (2.6). Then we have[13]

$$\begin{aligned}\tilde{q}^{(4)}_{2m+1}(x) &= p_{2m+1}(x)\\ \tilde{q}^{(4)}_{2m}(x) &= 2^{2m} m!\,(a/2+m)! \sum_{j=0}^{m} \frac{1}{2^{2j} j!\,(a/2+j)!}\, p_{2j}(x)\\ \tilde{r}^{(4)}_{n} &= \frac{1}{2}\Gamma(2n+2)\,\Gamma(a+2n+2)\end{aligned} \tag{3.10}$$

Jacobi Case. In the Jacobi case

$$e^{-V(x)} = (1-x)^{a/2}\,(1+x)^{b/2} \tag{3.11}$$

The monic orthogonal polynomials and normalizations with respect to (2.16) are then

$$p_k(x) = 2^k k! \, \frac{\Gamma(a+b+k+1)}{\Gamma(a+b+2k+1)} \, P_k^{(a,\,b)}(x)$$

$$(p_k,\, p_k)_2 = 2^{(a+b+1+2k)} \, \frac{\Gamma(k+1)\, \Gamma(a+b+k+1)\, \Gamma(a+1+k)\, \Gamma(b+1+k)}{\Gamma(a+b+2k+2)\, \Gamma(a+b+2k+1)} \tag{3.12}$$

With

$$e^{-V(x)} \mapsto e^{-V(x)-(1/2)\log f(x)} = (1-x)^{(a-1)/2}\,(1+x)^{(b-1)/2} \tag{3.13}$$

in (2.5) the skew orthogonal polynomials and corresponding normalization at $\beta = 1$ are[13]

$$\tilde{q}^{(1)}_{2m}(x) = p_{2m}(x)$$

$$\begin{aligned}
\tilde{q}^{(1)}_{2m+1}(x) = 2^{2m+1}(2m)! \, &\frac{\Gamma(a+b+2m-1)}{\Gamma(a+b+4m+1)} \\
&\times \Big\{ (2m+1)(a+b+2m-1)\, P_{2m+1}^{(a-1,\,b-1)}(x) \\
&- \frac{(a+b-2)(a+2m)(b+2m)}{(a+b+4m)(a+b+4m+2)} \, \frac{d}{dx} \, P_{2m}^{(a-1,\,b-1)}(x) \Big\} \\
&= p_{2m+1}(x) + 2(2m+1)\, \frac{a-b}{4m+a+b+2}\, p_{2m}(x) \\
&- 8m \, \frac{(2m+a)(2m+b)(a+b+2m)}{\begin{pmatrix}(a+b+4m-1)(a+b+4m)(a+b+4m+1) \\ \times\,(a+b+4m+2)\end{pmatrix}} \, p_{2m-1}(x)
\end{aligned}$$

$$\tilde{r}^{(1)}_n = 2^{a+b+4n+2}(2n)! \, \frac{\Gamma(a+2n+1)\, \Gamma(b+2n+1)\, \Gamma(a+b+2n+1)}{\Gamma(a+b+4n+1)\, \Gamma(a+b+4n+3)} \tag{3.14}$$

where in obtaining the second equality in the formula for $\tilde{q}_{2m+1}(x)$ use has been made of the formulas

$$\frac{d}{dx} \, P_n^{(\alpha,\,\beta)}(x) = \frac{1}{2} \, P_{n-1}^{(\alpha+1,\,\beta+1)}(x)$$

$$(2n+\alpha+\beta+1)\, P_n^{(\alpha,\beta)}(x)$$
$$= \frac{(n+\alpha+\beta+1)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+2)}\, P_n^{(\alpha+1,\beta+1)}(x)$$
$$+\left(-\frac{(n+\alpha+\beta+1)(n+\beta+1)}{(2n+\alpha+\beta+2)}+\frac{(n+\alpha+\beta+1)(n+\alpha)}{(2n+\alpha+\beta)}\right)$$
$$\times P_{n-1}^{(\alpha+1,\beta+1)}(x) - \frac{(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)}\, P_{n-2}^{(\alpha+1,\beta+1)}$$

The non-uniqueness of the skew orthogonal polynomials up to the transformation (2.4) implies we can equally as well write $\tilde{q}_{2m+1}(x)$ in (3.14) as

$$\tilde{q}_{2m+1}(x) = p_{2m+1}(x) - 8m\, \frac{(2m+a)(2m+b)(a+b+2m)}{\begin{pmatrix}(a+b+4m-1)(a+b+4m)\\ \times(a+b+4m+1)(a+b+4m+2)\end{pmatrix}} \times p_{2m-1}(x) \tag{3.15}$$

At $\beta=4$, make the replacement

$$e^{-2V(x)} \mapsto e^{-2V(x)+\log f(x))} = (1-x)^{a+1}\,(1+x)^{b+1} \tag{3.16}$$

in (2.6). The corresponding skew orthogonal polynomials and normalizations are then[13]

$$\tilde{q}^{(4)}_{2m+1}(x) = p_{2m+1}(x)$$
$$\tilde{q}^{(4)}_{2m}(x) = \frac{2^{6m+a+b}}{\sqrt{\pi}}\, \frac{m!\, \Gamma(a/2+b/2+m+1)\, \Gamma(a/2+m+1)\, \Gamma(b/2+m+1)}{\Gamma(a+b+4m+2)}$$
$$\times \sum_{j=0}^{m} \frac{1}{j!\, 2^{4j}}\, \frac{\Gamma((a/2+b/2+j+1)}{\Gamma(a/2+j+1)\, \Gamma(b/2+j+1)}\, \frac{\Gamma(a+b+4j+2)}{\Gamma(a+b+2j+1)}\, p_{2j}(x)$$
$$= 2^{6m} m!\, \frac{\Gamma(a/2+b/2+m+1)\, \Gamma(a/2+m+1)\, \Gamma(b/2+m+1)}{\Gamma(a+b+4m+2)}$$
$$\times \sum_{j=0}^{m} \frac{1}{j!\, 2^{6j}}\, \frac{\Gamma(a+b+4j+2)}{\begin{pmatrix}\Gamma(a/2+b/2+j+1)\, \Gamma(a/2+j+1)\\ \times\Gamma(b/2+j+1)\end{pmatrix}}\, p_{2j}(x)$$
$$\tilde{r}^{(4)}_n = \frac{\begin{pmatrix}2^{a+b+4n+2}\Gamma(2n+2)\, \Gamma(a+2n+2)\\ \times\Gamma(b+2n+2)\, \Gamma(a+b+2n+2)\end{pmatrix}}{\Gamma(a+b+4n+2)\, \Gamma(a+b+4n+4)} \tag{3.17}$$

where in obtaining the second equality in the formula for $\tilde{q}^{(4)}_{2m}(x)$ use has been made of the duplication formula

$$\Gamma(z)\,\Gamma(z+1/2)=2^{1/2-2z}(2\pi)^{1/2}\,\Gamma(2z)$$

3.2. Derivation from General Formalism

$\beta=1$

The expression for skew orthogonal polynomials at $\beta=1$ is determined by solving the equation (2.25) for $Q^{(1)}=[\,Q_{jk}\,]_{j,\,k=0,\dots,\,N-1}$. In fact this can be done by premultiplying both sides by $(Q^{(1)\,T})^{-1}$ and equating the lower triangular entries on both sides (not including the diagonal). Using (2.3) we see that on the l.h.s.

$$(\tilde{R}^{-1}Q^{(1)})_- = \begin{pmatrix} * & & & & & & \\ 1/\tilde{r}^{(1)}_0 & * & & & & & \\ -Q_{30}/\tilde{r}^{(1)}_1 & -Q_{31}/\tilde{r}^{(1)}_1 & * & & & & \\ Q_{20}/\tilde{r}^{(1)}_1 & Q_{21}/\tilde{r}^{(1)}_1 & 1/\tilde{r}^{(1)}_1 & * & & & \\ -Q_{50}/\tilde{r}^{(1)}_2 & -Q_{51}/\tilde{r}^{(1)}_2 & -Q_{52}/\tilde{r}^{(1)}_2 & -Q_{53}/\tilde{r}^{(1)}_2 & * & & \\ Q_{40}/\tilde{r}^{(1)}_2 & Q_{41}/\tilde{r}^{(1)}_2 & Q_{42}/\tilde{r}^{(1)}_2 & Q_{43}/\tilde{r}^{(1)}_2 & 1/\tilde{r}^{(1)}_2 & * & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix}_{2N\times 2N} \tag{3.18}$$

where the subscript in ( )$_-$ denotes the strictly lower triangular entries. On the r.h.s. we note from the fact that $(Q^{(1)\,T})^{-1}$ is an upper triangular matrix with 1's along the diagonal and the explicit form (2.18) of $\mathcal{N}$ that

$$\begin{aligned} -((Q^{(1)\,T})^{-1}\,D^{-1}\mathcal{N}D'^{\,-1})_- &= -(D^{-1}\mathcal{N}D'^{\,-1})_- \\ &= \begin{pmatrix} * & & & & \\ \gamma_0 & * & & & \\ 0 & \gamma_1 & * & & \\ 0 & 0 & \gamma_2 & * & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix} \end{aligned} \tag{3.19}$$

where

$$\gamma_j := c_j/(p_{j+1},\,p_{j+1})_2\,(p_j,\,p_j)_2 \tag{3.20}$$

Equating (3.18) and (3.19) gives

$$\begin{aligned} \gamma_{2p} &= 1/\tilde{r}_p^{(1)} \qquad (p=0,..., N-1) \\ Q_{2p,\, l} &= 0 \qquad (l=0,..., 2p-1) \\ Q_{2p+1,\, l} &= 0 \qquad (l=0,..., 2p-2) \\ Q_{2p+1,\, 2p-1} &= -\gamma_{2p-1}\tilde{r}_p^{(1)} = -\gamma_{2p-1}/\gamma_{2p} \end{aligned} \tag{3.21}$$

while $Q_{2p+1,\, 2p}$ is left unspecified. Hence

$$\begin{aligned} \tilde{q}_{2j}^{(1)}(x) &= p_{2j}^{(1)}(x) \\ \tilde{q}_{2j+1}^{(1)}(x) &= p_{2j+1}^{(1)}(x) + Q_{2p+1,\, 2p} p_{2j}^{(1)}(x) - \frac{\gamma_{2j-1}}{\gamma_{2j}}\, p_{2j-1}^{(1)}(x) \\ \tilde{r}_p^{(1)} &= -1/\gamma_{2p} \end{aligned} \tag{3.22}$$

Note that the fact that $Q_{2p+1,\, 2p}$ is arbitrary in the formula for $\tilde{q}_{2j+1}^{(1)}(x)$ is consistent with the skew orthogonal polynomials being non-unique up to the transformation (2.4). The simplest choice is $Q_{2p+1,\, 2p}=0$.

Let us verify that (3.22) with $Q_{2p+1,\, 2p}=0$ reclaims the results (3.1), (3.6) and (3.14). First γ_j must be specified (recall (3.20)). From (2.19) and (3.20) we have that

$$\mathbf{n} p_k(x) = -\gamma_k(p_k, p_k)_2\, p_{k+1}(x) + \text{lower degree term} \tag{3.23}$$

which together with the explicit form of $\mathbf{n}$ for each of the classical weight functions and the fact that each $p_k(x)$ is monic implies

$$\gamma_k(p_k, p_k)_2 = \begin{cases} 1, & \text{Hermite} \\ \frac{1}{2}, & \text{Laguerre} \\ \frac{1}{2}(2k+2+a+b), & \text{Jacobi} \end{cases} \tag{3.24}$$

Furthermore, the explicit form of $(p_k, p_k)_2$ is given by (3.1), (3.4) and (3.12) in the Hermite, Laguerre and Jacobi cases respectively. A straightforward calculation then enables the formulas of the previous section for $\beta=1$ to be reclaimed from the general formulas (3.22).

$\beta=4$

At $\beta=4$ we must solve the equation (2.27). This we will do for the matrix $(Q^{(4)})^{-1} := L = [\tilde{\beta}_{jk}^{(4)}]_{j,\, k=0,..., N-1}$, where L is lower triangular with

1's along the diagonal. In this case it is sufficient to multiply both sides of (2.27) by $Q^{(4)\,T}$ and equate the strictly lower triangular parts of both sides. On the l.h.s. we have

$$(L\tilde{R}^{(4)})_- = \begin{pmatrix} * & & & & & \\ -\tilde{r}_0^{(4)} & * & & & & \\ -\tilde{r}_0^{(4)}\tilde{\beta}_{21}^{(4)} & \tilde{r}_0^{(4)}\tilde{\beta}_{20}^{(4)} & * & & & \\ -\tilde{r}_1^{(4)}\tilde{\beta}_{31}^{(4)} & \tilde{r}_1^{(4)}\tilde{\beta}_{30}^{(4)} & -\tilde{r}_1^{(4)} & * & & \\ -\tilde{r}_1^{(4)}\tilde{\beta}_{41}^{(4)} & \tilde{r}_1^{(4)}\tilde{\beta}_{40}^{(4)} & -\tilde{r}_1^{(4)}\tilde{\beta}_{43}^{(4)} & \tilde{r}_1^{(4)}\tilde{\beta}_{42}^{(4)} & * & \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix}_{2N\times 2N} \tag{3.25}$$

while on the r.h.s.

$$(\mathcal{N} Q^{(4)\,T})_- = (\mathcal{N})_- \tag{3.26}$$

Equating (3.25) and (3.26) gives

$$\begin{aligned} c_{2p} &= \tilde{r}_p^{(4)} \qquad p = 0,..., N-1 \\ \tilde{\beta}_{2p+1,\,j}^{(4)} &= 0 \qquad j = 0,..., 2p-1 \\ \tilde{\beta}_{2p,\,j}^{(4)} &= 0 \qquad j = 0,..., 2p-3,\ j = 2p-1 \\ \tilde{\beta}_{2p,\,2p-2}^{(4)} &= -c_{2p-1}/\tilde{r}_{p-1}^{(4)} = -c_{2p-1}/c_{2p-2} \end{aligned} \tag{3.27}$$

while $\tilde{\beta}_{2p+1,\,2p}^{(4)}$ is undetermined. Thus we have

$$\begin{aligned} p_{2j+1}(x) &= \tilde{q}_{2j+1}^{(4)}(x) + \tilde{\beta}_{2j+1,\,2j}^{(4)}\tilde{q}_{2j}^{(4)}(x) \\ p_{2j}(x) &= \tilde{q}_{2j}^{(4)}(x) - \frac{c_{2j-1}}{c_{2j-2}}\,\tilde{q}_{2j-2}^{(4)}(x) \\ \tilde{r}_p^{(4)} &= c_{2p} \end{aligned} \tag{3.28}$$

The non-uniqueness of the skew orthogonal polynomials up to the transformation (2.4) implies we can choose $\tilde{\beta}_{2j+1,\,2j}^{(4)} = 0$. Doing this, and solving for $\tilde{q}_{2j}^{(4)}(x)$ in the second equation of (3.28) gives

$$\begin{aligned} \tilde{q}_{2j+1}^{(4)}(x) &= p_{2j+1}(x) \\ \tilde{q}_{2j}^{(4)}(x) &= \left(\prod_{p=0}^{j-1} \frac{c_{2p+1}}{c_{2p}}\right) \sum_{l=0}^{j} \left(\prod_{p=0}^{l-1} \frac{c_{2p+1}}{c_{2p}}\right)^{-1} p_{2l}(x) \\ \tilde{r}_p^{(4)} &= c_{2p} \end{aligned} \tag{3.29}$$

We recall that the quantities c_j are, according to (3.20) and (3.24), simply related to the norm $(p_{j+1}, p_{j+1})_2$ which in turn is given by (3.1), (3.4) and (3.12) in the Hermite, Laguerre and Jacobi cases respectively. From this it is straightforward to verify that the general formula reproduces the results of the previous section for $\beta=4$.

4. SUMMATION FORMULAS

The results of Section 2.1 show that the n-point distributions are given in terms of the quantities $S_1(x, y)$ and $S_4(x, y)$ (recall (2.9) and (2.12)) at $\beta=1$ and $\beta=4$ respectively. It is of interest to compare these formulas with the expression for the n-point distribution (2.1) at $\beta=2$. With $w_2(x)=e^{-2V(x)}$ it is an easy consequence of (1.2) that

$$\begin{aligned} \rho_{(n)}(x_1,..., x_n) &= \det[S_2(x_j, x_k)]_{j,k=1,...,n} \\ S_2(x, y) &:= e^{-V(x)-V(y)} \sum_{l=0}^{N-1} \frac{p_l(x)\, p_l(y)}{(p_l, p_l)_2} \end{aligned} \tag{4.1}$$

Thus again the n-point function is completely determined by a summation. In fact for $\{p_j(x)\}$ a general set of monic orthogonal polynomials this sum can be evaluated exactly according to the Christoffel–Darboux formula

$$\sum_{l=0}^{N-1} \frac{p_l(x)\, p_l(y)}{(p_l, p_l)_2} = \frac{1}{(p_{N-1}, p_{N-1})_2} \frac{p_N(x)\, p_{N-1}(y) - p_{N-1}(x)\, p_N(y)}{x-y} \tag{4.2}$$

Here we seek an analogous summation formula for $S_1(x, y)$ and $S_4(x, y)$.

There is some recent literature on this problem. Using different methods, Widom[15] and Forrester *et al.*[16] have expressed $S_1(x, y)$ and $S_4(x, y)$ in terms of the Christoffel–Darboux summation (4.2) in which $\{p_j(x)\}$ are the monic orthogonal polynomials associated with the weight function $w_2(x)=e^{-2V(x)}$ plus a "correction" term. In general the correction term (which is given in a different form in the two different formalisms refs. 15 and 6) does not appear to have a closed form evaluation. Exceptions are the Hermite (H) and Laguerre (L) cases. Then the correction term factorizes as a function of x times a function of y, both of which can be computed exactly (the results in the Hermite case were known from earlier work; see e.g., ref. 9). Explicitly

$$\begin{aligned} S_1^{(H)}(x, y) = S_2^{(H)}(x, y) + \frac{e^{-y^2/2}}{2^{N+1}\sqrt{\pi}\,(N-1)!} H_{N-1}(y) \\ \times \int_{-\infty}^{\infty} \operatorname{sgn}(x-t)\, e^{-t^2/2} H_N(t)\, dt \end{aligned}$$

$$S_1^{(H)\,\text{odd}}(x,y) = S_1^{(H)}(x,y) + \frac{H_{N-1}(x)}{\int_{-\infty}^{\infty} e^{-t^2/2} H_{N-1}(t)\,dt}$$

$$2S_4^{(H)}(x,y) = S_2^{(H)}(x,y)\,|_{N\mapsto 2N} + e^{-y^2/2} H_{2N}(y)\,\frac{2^{-2N}}{\sqrt{\pi}\,\Gamma(2N)} \times \int_{-\infty}^{x} e^{-t^2/2} H_{2N-1}(t)\,dt$$

$$S_1^{(L)}(x,y) = S_2^{(L)}(x,y) + \frac{N!}{4\Gamma(N+a)}\,y^{a/2} e^{-y/2} \left(\frac{d}{dy} L_N^a(y)\right) \times \int_0^{\infty} \mathrm{sgn}(x-y)\, y^{a/2-1} e^{-y/2} (L_N^a(y) - L_{N-1}^a(y))\,dy$$

$$2S_4^{(L)}(x,y) = S_2^{(L)}(x,y)\,|_{N\mapsto 2N} + \frac{(2N)!\; y^{a/2} e^{-y/2}}{2\Gamma(2N+a)}\,\frac{L_{2N}^a(y) - L_{2N-1}^a(y)}{y} \times \int_0^{x} t^{a/2} e^{-t/2} \frac{d}{dt} L_{2N}^a(t)\,dt \tag{4.3}$$

One feature of these formulas is that the quantities $S_1(x, y)$ and $S_4(x, y)$, in which the skew orthogonal polynomials are defined with respect to the skew inner products (2.5) and (2.6) (and thus involve $e^{-V(x)}$ and $e^{-2V(x)}$ respectively) are expressed in terms of the weight function $e^{-2V(x)}$. However, we have seen in the previous section that it is the skew orthogonal polynomials at $\beta=1$ with the replacement

$$2V(x) \mapsto 2\tilde{V} = 2V(x) + \log f(x) \tag{4.4}$$

in (2.5), and the skew orthogonal polynomials at $\beta=4$ with the replacement

$$2V(x) \mapsto 2\tilde{V} = 2V(x) - \log f(x) \tag{4.5}$$

in (2.6) which are naturally expressed in terms of the orthogonal polynomials for the weight function $e^{-2V(x)}$. This suggests that we consider the quantities $\tilde{S}_1(x, y)$ and $\tilde{S}_4(x, y)$, defined as in (2.9) and (2.12) but with the corresponding skew orthogonal polynomials defined with the replacements (4.4) and (4.5) respectively, and seek to sum them in terms of the orthogonal polynomials for the weight function $e^{-2V(x)}$. In the Hermite case nothing changes because, according to the definition (2.14), $f(x)=1$ and so $\log f(x)=0$ in (4.4) and (4.5). However, in the Laguerre case this is already a different viewpoint as then $V(x)$ is modified by $a \mapsto a \pm 1$. Indeed we find that the simple structures (3.22) and (3.28) giving the skew orthogonal polynomials for the modified $V(x)$ in terms of the orthogonal polynomials for the weight function $e^{-2V(x)}$ allows $\tilde{S}_1(x, y)$ and $\tilde{S}_4(x, y)$ to be summed in the classical cases by the same general formula in each case.

$\beta = 1$, N **even**

Let us write

$$p_l(x) = \sum_{j=0}^{l} \tilde{\beta}_{lj}^{(1)} \tilde{q}_j^{(1)}(x), \qquad \tilde{\beta}_{ll}^{(1)} = 1 \tag{4.6}$$

where $\{\tilde{q}_j^{(1)}(x)\}_{j=0,1,\ldots}$ is a set of monic skew orthogonal polynomials with respect to the inner product (2.5) (modified so that V is replaced by $\tilde{V}$), and $\{p_l(x)\}_{l=0,1,\ldots}$ is the set of monic polynomials with respect to the inner product (2.16). The quantity $S_1(x, y)$ can be expressed in terms of the polynomials $p_l(x)$.

Proposition 4.1. Let N be even and consider in general independent potentials V and $\tilde{V}$. With $\tilde{S}_1(x, y)$ defined by (2.9), modified so that $V \mapsto \tilde{V}$ and $q_k^{(1)}(x) \mapsto \tilde{q}_k^{(1)}(x)$, and $\{p_j(x)\}_{j=0,1,\ldots}$ the set of monic orthogonal polynomials associated with the weight function $e^{-2V(x)}$ (assumed complete) we have

$$\tilde{S}_1(x, y) = e^{-2V(x) + \tilde{V}(x) - \tilde{V}(y)} \left(\sum_{n=0}^{N-1} \frac{p_n(x)\, p_n(y)}{(p_n, p_n)_2} \right.$$
$$\left. + \sum_{n=N}^{\infty} \sum_{k=0}^{N-1} \frac{p_n(x)}{(p_n, p_n)_2} \tilde{\beta}_{nk}^{(1)} \tilde{q}_k^{(1)}(y) \right) \tag{4.7}$$

Proof. Our derivation is motivated by the derivation of a similar formula in ref. 12. Now, from the definitions (2.8) and (2.5) (the latter with $V \mapsto \tilde{V}$) it is possible to write

$$\tilde{\Phi}_k(x) = e^{\tilde{V}(x)} \langle \tilde{q}_k^{(1)}(y) \mid \delta(x-y) \rangle_1$$

But from the completeness of $\{p_j(x)\}$ we can make the expansion

$$\delta(x-y) = e^{-2V(x)} \sum_{n=0}^{\infty} \frac{p_n(x)\, p_n(y)}{(p_n, p_n)_2} \tag{4.8}$$

Substituting (4.6) for $p_n(y)$ and using the skew orthogonality of $\{\tilde{q}_k(y)\}$ shows

$$\tilde{\Phi}_{2k}(x) = \tilde{r}_k^{(1)} e^{\tilde{V}(x)} e^{-2V(x)} \sum_{v=2k+1}^{\infty} \frac{p_v(x)}{(p_v, p_v)_2} \tilde{\beta}_{v2k+1}^{(1)} \tag{4.9}$$

$$\tilde{\Phi}_{2k+1}(x) = -\tilde{r}_k^{(1)} e^{\tilde{V}(x)} e^{-2V(x)} \sum_{v=2k}^{\infty} \frac{p_v(x)}{(p_v, p_v)_2} \tilde{\beta}_{v2k}^{(1)} \tag{4.10}$$

Next substitute these formulas in the definition (2.9). The stated formula follows after minor manipulation involving further use of (4.6).

The first sum in (4.7) is evaluated according to the Christoffel–Darboux formula (4.2). This applies for general monic orthogonal polynomials $\{p_j(x)\}$. To evaluate the second sum requires knowledge of the transition coefficients $\tilde{\beta}_{lj}^{(1)}$. In the case of the classical polynomials these coefficients can be determined from (3.22) provided $\tilde{V}$ is related to V by the right hand side of (4.4). Assuming this, setting $Q_{2p+1,\,2p} = 0$ and comparing with the definition (4.6) we see that

$$\tilde{\beta}_{2l,\,j}^{(1)} = 0, \qquad (j = 0,\dots,\, 2l-1)$$

$$\tilde{\beta}_{2l+1,\,2j}^{(1)} = 0, \qquad (j = 0,\dots,\, l)$$

$$\tilde{\beta}_{2l+1,\,2j+1}^{(1)} = \frac{\prod_{k=1}^{l} a_k}{\prod_{k=1}^{j} a_k}, \qquad a_k := \frac{\gamma_{2k-1}}{\gamma_{2k}}$$

In fact we don't require the explicit form of the $\tilde{\beta}_{lj}^{(1)}$, but rather their factorization property

$$\tilde{\beta}_{nk}^{(1)} = \tilde{\beta}_{n,\,N-1}^{(1)} \tilde{\beta}_{N-1,\,k}^{(1)}, \qquad n \geqslant N \tag{4.11}$$

which is evident from the above formulas. Substituting (4.11) in the double summation in (4.7) and recalling (4.6) shows

$$\begin{aligned}
&\sum_{n=N}^{\infty} \sum_{k=0}^{N-1} \frac{p_n(x)}{(p_n, p_n)_2} \tilde{\beta}_{nk}^{(1)} \tilde{q}_k(y) \\
&\quad = \left(\sum_{n=N}^{\infty} \frac{p_n(x)}{(p_n, p_n)_2} \tilde{\beta}_{n,\,N-1}^{(1)} \right) p_{N-1}(y) \\
&\quad = \left(\frac{1}{\tilde{r}_{N/2-1}^{(1)}} e^{-V(x)+2\tilde{V}(x)} \tilde{\Phi}_{N-2}(x) - \frac{p_{N-1}(x)}{(p_{N-1}, p_{N-1})_2} \right) p_{N-1}(y)
\end{aligned}$$

where the second equality follows from (4.9) and the fact that N is assumed even. Thus

$$\tilde{S}_1(x, y) = e^{-(V(x) - \tilde{V}(x))} e^{(V(y) - \tilde{V}(y))} S_2(x, y)|_{N \mapsto N-1} + \gamma_{N-2} e^{-\tilde{V}(y)} \tilde{\Phi}_{N-2}(x)\, p_{N-1}(y)$$

Making use of (3.22) in the definition (2.8) of $\tilde{\Phi}_{N-2}$ allows all reference to the skew orthogonal polynomials to be eliminated, and gives a summation formula of the same form for all the classical cases.

Proposition 4.2. For the classical weights and N even

$$\tilde{S}_1(x, y) = e^{-(V(x) - \tilde{V}(x))} e^{(V(y) - \tilde{V}(y))} S_2(x, y)|_{N \mapsto N-1} + \gamma_{N-2} e^{-\tilde{V}(y)} p_{N-1}(y)\, \tfrac{1}{2} \int_{-\infty}^{\infty} \mathrm{sgn}(x-t)\, p_{N-2}(t)\, e^{-\tilde{V}(t)}\, dt \tag{4.12}$$

where γ_{N-2} is specified in terms of $(p_{N-2}, p_{N-2})_2$ by (3.24) and V and $\tilde{V}$ are related by (4.4).

It is of interest to compare (4.12) with the summation formulas for $\tilde{S}_1$ in (4.3). In fact neither the Hermite nor Laguerre summations in (4.3) are of the same form as (4.12). But in both cases the equality of the different forms can be established directly. Consider (4.12) in the Hermite case when it reads

$$\tilde{S}_1^{(H)}(x, y) = S_2^{(H)}(x, y)|_{N \mapsto N-1} + \frac{e^{-y^2/2} H_{N-1}(y)}{2^N \pi^{1/2} (N-2)!} \times \int_{-\infty}^{\infty} dt\, \mathrm{sgn}(x-t)\, e^{-t^2/2} H_{N-2}(t)$$

This can be brought into agreement with the formula for $\tilde{S}_1^{(G)}$ in (4.3) if we note from the identity

$$H_{2k}(y) - \frac{1}{4(k+1/2)} H_{2k+2}(y) = \frac{1}{2k+1} e^{y^2/2} \frac{d}{dy} (e^{-y^2/2} H_{2k+1}(y))$$

that

$$\Phi_{2k}(x) - \frac{1}{k+1/2} \Phi_{2k+2}(x) = \frac{2^{-2k}}{2k+1} e^{-x^2/2} H_{2k+1}(x) \tag{4.13}$$

In the Laguerre case the manipulation of (4.12) into the form given in (4.3) is undertaken in the Appendix.

$\beta = 1$, N **odd**

In the case of N odd, we see from the definitions (2.11) that

$$\tilde{S}_1^{\mathrm{odd}}(x, y) = \tilde{S}_1(x, y)|_{N \mapsto N-1} + \frac{\tilde{q}_{N-1}^{(1)}(y)}{2\tilde{s}_{N-1}}$$
$$+ \frac{\tilde{\Phi}_{N-1}(x)}{\tilde{s}_{N-1}} \sum_{k=0}^{(N-1)/2-1} \frac{e^{-\tilde{V}(y)}}{\tilde{r}_k^{(1)}} (-\tilde{s}_{2k}\tilde{q}_{2k+1}^{(1)}(y) + \tilde{s}_{2k+1}\tilde{q}_{2k}^{(1)}(y))$$
$$- \frac{\tilde{q}_{N-1}^{(1)}(y)}{\tilde{s}_{N-1}} \sum_{k=0}^{(N-1)/2-1} \frac{e^{-\tilde{V}(y)}}{\tilde{r}_k^{(1)}} (-\tilde{s}_{2k}\tilde{\Phi}_{2k+1}(x) + \tilde{s}_{2k+1}\tilde{\Phi}_{2k}(x))$$

where

$$\tilde{s}_k := \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-\tilde{V}(x)} \tilde{q}_k^{(1)}(x)\, dx \tag{4.14}$$

The quantity $\tilde{S}_1(x, y)|_{N \mapsto N-1}$ is evaluated by (4.12). Furthermore, from the definitions we see that

$$\sum_{k=0}^{(N-1)/2-1} \frac{e^{-\tilde{V}(y)}}{\tilde{r}_k^{(1)}} (-\tilde{s}_{2k}\tilde{q}_{2k+1}^{(1)}(y) + \tilde{s}_{2k+1}\tilde{q}_{2k}^{(1)}(y))$$
$$= -\lim_{x \to \infty} \tilde{S}_1(x, y)|_{N \mapsto N-1} = -\gamma_{N-3}\tilde{s}_{N-3} e^{-\tilde{V}(y)} p_{N-2}(y) \tag{4.15}$$

where here the second equality follows from (4.12), while

$$\sum_{k=0}^{(N-1)/2-1} \frac{1}{\tilde{r}_k^{(1)}} (-\tilde{s}_{2k}\tilde{\Phi}_{2k+1}(x) + \tilde{s}_{2k+1}\tilde{\Phi}_{2k}(x))$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{sgn}(x-y) \Big(-\lim_{x' \to \infty} \tilde{S}_1(x', y)|_{N \mapsto N-1}\Big)\, dy$$
$$= -\gamma_{N-3}\tilde{s}_{N-3}\tilde{\phi}_{N-2}(x) \tag{4.16}$$

where here the second equality follows from (4.15) and

$$\tilde{\phi}_j(x) := \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-\tilde{V}(y)}\, \mathrm{sgn}(x-y)\, p_j(y)\, dy \tag{4.17}$$

Noting from (3.22) that for N odd, $\tilde{q}_{N-1}(y)=p_{N-1}(y)$, allows the skew orthogonal polynomial to be eliminated from (4.14), and thus a summation formula for $\tilde{S}_1^{\text{odd}}(x,y)$ which involves only the orthogonal polynomials.

Proposition 4.3. For the classical cases and N odd

$$\tilde{S}_1^{\text{odd}}(x,y)=\tilde{S}_1(x,y)\,|_{N\mapsto N-1}+\frac{p_{N-1}(y)}{2\tilde{s}_{N-1}}$$
$$-\gamma_{N-3}\tilde{s}_{N-3}\frac{e^{-\tilde{V}(y)}}{\tilde{s}_{N-1}}\,(\tilde{\phi}_{N-1}(x)\,p_{N-2}(y)-p_{N-1}(y)\,\tilde{\phi}_{N-2}(x)) \tag{4.18}$$

where $\tilde{\phi}_j$ is specified by (4.17), $\tilde{s}_k$ (k even) by (4.14) with $\tilde{q}_k^{(1)}(x)\mapsto p_k(x)$ and the quantities γ_{N-3}, V and $\tilde{V}$ as in Proposition 4.2.

Let us show how to write (4.18) in the Hermite case as presented in (4.3). Now, from the definite integral

$$\int_{-\infty}^{\infty} e^{-x^2/2}H_n(x)\,dx=(2\pi)^{1/2}\,2^{n/2}(n-1)(n-3)\cdots 3\cdot 1 \qquad n \text{ even}$$

we see that

$$\gamma_{N-3}\frac{\tilde{s}_{N-3}}{\tilde{s}_{N-1}}=2\,\frac{\gamma_{N-3}}{N-2}=\gamma_{N-2} \tag{4.19}$$

This fact shows that the final term in the second line of (4.18) is equal to the final term in the formula (4.12) for $\tilde{S}_1(x,y)$. Also, making use of the first equality in (4.19) and (4.13) shows

$$e^{-\tilde{V}(y)}p_{N-2}(y)\,\gamma_{N-3}\left(\tilde{\phi}_{N-3}(x)-\frac{s_{N-3}}{s_{N-1}}\,\tilde{\phi}_{N-1}(x)\right)$$
$$=e^{-\tilde{V}(x)-\tilde{V}(y)}\,\frac{p_{N-2}(x)\,p_{N-2}(y)}{(p_{N-2},p_{N-2})_2}$$

We see from (4.12) with $N\mapsto N-1$ that this identity provides the final step in identifying (4.18) in the Hermite case with the formula in (4.3).

$\beta=4$

The strategy for obtaining a summation formula at $\beta=4$ is similar to that used at $\beta=1$. We write

$$p_l(x)=\sum_{j=0}^{l}\tilde{\beta}_{lj}^{(4)}\,\tilde{q}_j(x), \qquad \tilde{\beta}_{ll}^{(4)}=1 \tag{4.20}$$

where $\{\tilde{q}_j^{(4)}(x)\}_{j=0,1,\dots}$ is a set of monic skew orthogonal polynomials with respect to the inner product (2.6) (modified so that V is replaced by $\tilde{V}$) and $\{p_l(x)\}_{l=0,1,\dots}$ is the set of monic polynomials with respect to the inner product (2.16).

Proposition 4.4. Let V and $\tilde{V}$ be in general independent potentials. With $\tilde{S}_4(x, y)$ defined by (2.12), modified so that $V \mapsto \tilde{V}$ and $q_k^{(4)}(x) \mapsto \tilde{q}_k^{(4)}(x)$, and $\{p_j(x)\}_{j=0,1,\dots}$ the set of monic orthogonal polynomials associated with the weight function $e^{-2V(x)}$ (assumed complete) we have

$$\tilde{S}_4(x, y) = \frac{1}{2} e^{-2V(y) - \tilde{V}(x) + \tilde{V}(y)} \left(\sum_{n=0}^{2N-1} \frac{p_n(x)\, p_n(y)}{(p_n, p_n)_2} + \sum_{n=2N}^{\infty} \sum_{k=0}^{2N-1} \frac{p_n(x)}{(p_n, p_n)_2} \tilde{\beta}_{nk}^{(4)} \tilde{q}_k(x) \right) \tag{4.21}$$

Proof. Again our derivation is motivated by the workings in ref. 12. From the fact that

$$\frac{d}{dx}\left(e^{-\tilde{V}(x)} \tilde{q}_{2m}^{(4)}(x)\right) = \frac{e^{\tilde{V}(x)}}{2} \left(e^{-2\tilde{V}(x)} \frac{d}{dx} \tilde{q}_{2m}^{(4)}(x) + \frac{d}{dx}\left(e^{-2\tilde{V}(x)} \tilde{q}_{2m}^{(4)}(x)\right) \right)$$

we can check from the definition (2.6) that it's possible to write

$$\frac{d}{dx}\left(e^{-\tilde{V}(x)} \tilde{q}_{2m}^{(4)}(x)\right) = e^{\tilde{V}(x)} \langle \delta(x-y) \mid \tilde{q}_{2m}^{(4)}(y) \rangle_4$$

Substituting (4.8) for $\delta(x-y)$, and then substituting (4.20) for $p_j(y)$ and making use of the skew orthogonality of $\{q_j^{(4)}(x)\}$ shows

$$\frac{d}{dx}\left(e^{-\tilde{V}(x)} \tilde{q}_{2m}^{(4)}(x)\right) = -\tilde{r}_m^{(4)} e^{\tilde{V}(x) - 2V(x)} \sum_{\nu=2m+1}^{\infty} \frac{p_\nu(x)}{(p_\nu, p_\nu)_2} \tilde{\beta}_{\nu, 2m+1}^{(4)} \tag{4.22}$$

Proceeding similarly we can also show

$$\frac{d}{dx}\left(e^{-\tilde{V}(x)} \tilde{q}_{2m+1}^{(4)}(x)\right) = \tilde{r}_m^{(4)} e^{\tilde{V}(x) - 2V(x)} \sum_{\nu=2m}^{\infty} \frac{p_\nu(x)}{(p_\nu, p_\nu)_2} \tilde{\beta}_{\nu, 2m}^{(4)} \tag{4.23}$$

Apart from the sign, these equations are formally the same as those in (4.9) and (4.10). The stated formula thus follows as in the derivation of (4.7).

Since the first sum is evaluated by the Christoffel–Darboux formula (4.2), it remains to evaluate the second sum in (4.21). This in turn requires the value of the $\beta_{nk}^{(4)}$. Now, according to (3.28) with $\tilde{\beta}_{2p+1,\,2p}^{(4)}=0$, in the classical cases with V and $\tilde{V}$ related by (4.5) the only non-zero value of $\tilde{\beta}_{nk}^{(4)}$ for $n>k$ is

$$\tilde{\beta}_{2n,\,2n-2}^{(4)} = -\frac{c_{2n-1}}{c_{2n}}$$

where c_n is specified by (3.20) and (3.24). Hence in the classical cases

$$\tilde{S}_4(x,y) = \frac{1}{2}\, e^{-(V(y)-\tilde{V}(y))} e^{V(x)-\tilde{V}(x)} S_2(x,y)\,|_{N\mapsto 2N} + \frac{1}{2}\, e^{-\tilde{V}(x)+\tilde{V}(y)-2V(y)} \frac{c_{2N-1}}{c_{2N}} \frac{p_{2N}(y)}{(p_{2N}, p_{2N})_2}\, \tilde{q}_{2N-2}^{(4)}(x) \tag{4.24}$$

We note that reference to the skew-orthogonal polynomial $\tilde{q}_{2N-2}^{(4)}(x)$ can be eliminated by noting from (4.22), and the fact that the only non-zero value of $\tilde{\beta}_{\nu,\,2m+1}^{(4)}$ is $\tilde{\beta}_{2m+1,\,2m+1}^{(4)}=1$, that

$$e^{-\tilde{V}(x)}\tilde{q}_{2N-2}(x) = -\frac{\tilde{r}_{N-1}^{(4)}}{(p_{2N-1}, p_{2N-1})_2} \int_x^\infty dt\, e^{-2V(t)+\tilde{V}(t)} p_{2N-1}(t) \tag{4.25}$$

Using (3.28) to eliminate $\tilde{r}_{N-1}^{(4)}$ and recalling (3.20), this formula substituted in (4.24) gives the summation of $\tilde{S}_4(x,y)$.

Proposition 4.5. For the classical weights with V and $\tilde{V}$ related by (4.5)

$$\tilde{S}_4(x,y) = \tfrac{1}{2} e^{-(V(y)-\tilde{V}(y))} e^{V(x)-\tilde{V}(x)} S_2(x,y)\,|_{N\mapsto 2N} - \tfrac{1}{2} e^{\tilde{V}(y)-2V(y)} \gamma_{2N-1} p_{2N}(y) \int_x^\infty dt\, e^{-2V(t)+\tilde{V}(t)} p_{2N-1}(t) \tag{4.26}$$

In the Hermite case (4.26) gives immediate agreement with the formula in (4.3). However in the Laguerre case (4.26) has a different structure to the formula in (4.3). The manipulations necessary to show that the two formulas do indeed agree are undertaken in the Appendix.

5. SUMMARY

Classical skew orthogonal polynomials occur in the calculation of the n-point distribution function $\rho_{(n)}$ associated with the eigenvalue p.d.f. (1.3) with $w_\beta(x)$ a classical weight function and $\beta=1$ and 4. In particular they occur in a certain sum, denoted $S_\beta(x, y)$, which determines $\rho_{(n)}$. This sum is specified in (2.9) for $\beta=1$ and N even, in (2.11) for $\beta=1$ and N odd and in (2.12) for $\beta=4$. The main achievement of this paper has been the closed form evaluation of $S_\beta(x, y)$ in terms of particular classical orthogonal polynomials naturally related to the classical skew orthogonal polynomials. A self contained presentation of the relationship between the classical orthogonal and skew orthogonal polynomials is undertaken in Section 3. The summation formulas for $\tilde{S}_\beta(x, y)$ (the tilde on $\tilde{S}_\beta$ denotes that the weight functions are modified according to (4.4) and (4.5)), which apply equally as well to all the classical cases are given by (4.12) for $\beta=1$ and N even, (4.18) for $\beta=1$ and N odd and (4.26) for $\beta=4$.

APPENDIX

Here the summation formulas of (4.3) for $S_\beta(x, y)$ in the Laguerre case will be shown to give agreement with expressions implied by the general formulas (4.12) with the substitutions (3.24) and (3.4). Consider first the case $\beta=1$. Because $\tilde{S}_1(x, y)$ is defined with the replacement (4.4) and $f(x)=x$ in the Laguerre case, we see that we are required to show

$$S_1(x, y) = \tilde{S}_1(x, y)\,|_{a\mapsto a+1} \tag{A.1}$$

We see from (4.12) and (3.4) that the r.h.s. is presented as a series in the linearly independent set of functions $\{y^{a/2}e^{-y/2}L_k^{a+1}(y)\}$. On the other hand, the l.h.s. can also be expressed in terms of this basis by using the first identity in (3.7) (with $a\mapsto a+1$). Thus we must show that the coefficients of each $y^{a/2}e^{-y/2}L_k^{a+1}(y)$ (which are functions of x) agree. The largest value of k which occurs is $k=N-1$. On the l.h.s. the coefficient is then

$$x^{a/2}e^{-x/2}\frac{(N-1)!}{\Gamma(a+N)}L_{N-1}^a(x)$$
$$-\frac{N!}{4\Gamma(a+N)}\int_0^\infty \operatorname{sgn}(x-u)\,u^{a/2-1}e^{-u/2}L_N^{a-1}(u)\,du \tag{A.2}$$

while on the r.h.s. the coefficient is given by

$$-\frac{(N-1)!}{4\Gamma(a+N)}\int_0^\infty \operatorname{sgn}(x-u)\,L_{N-2}^{a+1}(u)\,u^{a/2}e^{-u/2}\,du \tag{A.3}$$

Now, by making use of the formula[11]

$$\int_0^\infty x^{\alpha/2} x^{-x/2} L_n^{\alpha+1}(x)\, dx = \begin{cases} \dfrac{\Gamma((n+3)/2))\, \Gamma(\alpha+n+2)}{2^{\alpha/2-1}\Gamma(n+2)\, \Gamma((n+\alpha+3)/2))}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \tag{A.4}$$

we can check that both (A.2) and (A.3) agree in the limit $x \to \infty$. It thus suffices to show that both expressions have the same derivative. This we can do by making use of the identities (3.7) as well as the further identity

$$x L_{N-1}^{a+1}(x) = -N L_n^{a+1}(x) + (2N+a)\, L_{N-1}^{a+1}(x) - (N+a)\, L_{N-2}^{a+1}(x) \tag{A.5}$$

The coefficient of $y^{a/2} e^{-y/2} L_k^{a+1}(y)$ for $k < N$ on the l.h.s. of (A.1) is given by

$$\left(\frac{k!}{\Gamma(a+1+k)}\, L_k^a(x) - \frac{(k+1)!}{\Gamma(a+2+k)}\, L_{k+1}^a(x) \right) x^{a/2} e^{-x/2} \tag{A.6}$$

while on the r.h.s. it is given by

$$\left(\frac{k!}{\Gamma(a+2+k)}\, x L_k^{a+1}(x) \right) x^{a/2} e^{-x/2} \tag{A.7}$$

Use of (A.5) and the first identity in (3.7) shows that these expressions agree.

A similar procedure can be adopted to show that the formula in (4.3) for $S_4(x, y)$ in the Laguerre case agrees with that implied by (4.26). Specifically, the task is to show

$$S_4(x, y) = \tilde{S}_4(x, y)\,|_{a \mapsto a-1} \tag{A.8}$$

This we do by comparing coefficients of $y^{a/2-1} e^{-y/2} L_k^{a-1}(y)$, $k = 0,..., 2N$. In the case $k = 2N$ on the l.h.s. use of (A.5) shows this coefficient is equal to

$$\frac{1}{2}\, x^{a/2} e^{-x/2}\, \frac{(2N-1)!^2\, L_{2N-1}^a(x)}{\Gamma(2N)\, \Gamma(a+2N)} (-2N)$$
$$+ \frac{(2N)!}{4\Gamma(2N+a)} \int_0^x t^{a/2} e^{-t/2} \left(\frac{d}{dt}\, L_{2N}^a(t) \right) dt$$

while on the r.h.s. we read off from (4.26), (3.24) and (3.4) that the same coefficient is

$$\frac{(2N)!}{4\Gamma(a+2N-1)} \int_x^\infty t^{a/2-1} e^{-t/2} L_{2N-1}^{a-1}(t)\, dt$$

One now checks that both the expressions vanish as $x \to 0$ (for the latter this requires (A.4)), and then verifies the equality of the derivatives. Equating coefficients of $y^{a/2-1} e^{-y/2} L_k^a(y)$ for $k < 2N$ gives an identity equivalent to the first formula in (3.7), thus verifying their equality.

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